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Vibration and buckling of a double-beam system under compressive axial loading

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Abstract

On the basis of the Bernoulli–Euler beam theory, the properties of free transverse vibration and buckling of a double-beam system under compressive axial loading are investigated in this paper. It is assumed that the two beams of the system are simply supported and continuously joined by a Winkler elastic layer. Explicit expressions are derived for the natural frequencies and the associated amplitude ratios of the two beams, and the analytical solution of the critical buckling load is obtained. The influences of the compressive axial loading on the responses of the double-beam system are discussed. It is shown that the critical buckling load of the system is related to the axial compression ratio of the two beams and the Winkler elastic layer, and the properties of free transverse vibration of the system greatly depend on the axial compressions.

1. Introduction

Beam-type structures are broadly adopted in civil, mechanical, and aerospace engineering. Therefore, the vibration and buckling problems of single beam and beam systems are of considerable practical interest and have wide application in engineering practice. In the past few decades, much attention has been drawn to the vibration and buckling of single one-dimensional continuous systems such as beams [1–8].

An important technological extension of the concept of the single beam is that of the double-beam system such as double-beam cranes, double-beam spectrometers, double-beam interferometers, etc. As a complex continuous system consisting of two one-dimensional solids joined by elastic medium, the elastically connected double-beam system has attracted great interest and its different aspects of dynamics have been investigated [9–14]. Seelig and Hoppmann II [15] presented the development and solution of the differential equations of motion of an elastically connected double-beam system subjected to an impulsive load. Rao [16] considered the free response of Timoshenko beam systems. Oniszczuk [17] discussed free transverse vibrations of two simply supported Bernoulli–Euler beams connected by a Winkler elastic layer. However, without considering

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the effect of axial loading these studies are limited to the cases with negligible axial loads. As a matter of fact, the phenomenon of transverse vibration of beams under axial compression often occurs in the aerospace, civil, and mechanical industries. For instance, in the design of certain spacecraft structural components it sometimes becomes necessary to determine the natural frequencies and mode shapes of beam-type components, which are in a state of preload or prestress. In order to understand this phenomenon, some studies regarding single beam systems with axial loads has been conducted in the past [18–23].

As an extension of the work of Oniszczuk [17], which does not consider the axial force, the free vibration and buckling of a double-beam system under axial loading are studied in the present paper. It is assumed that the system under consideration is composed of two parallel, slender, prismatic, and homogeneous beams continuously joined by a Winkler elastic layer. Both beams have the same length. It is also supposed that the buckling can only occur in the plane where the double-beam system lies. As the general vibration and buckling analysis of an elastically connected double-beam system is quite laborious due to the large variety of possible combinations of the boundary conditions, the discussions are limited only to the case of simply supported beams. The explicit expressions are derived for natural frequencies and associated amplitude ratios of the two beams, and the analytical solution of the critical buckling load is obtained. The effects of axial loading on the responses of the system are investigated.

2. Formulation

The Bernoulli–Euler beam theory is adopted in this study. This theory is on the basis of the assumption that plane cross-sections of a beam remain plane during flexure and that the radius of curvature of a bent beam is large compared with the beam's depth. It is valid only if the ratio of the depth to the length of the beam is small and the beams are excited at low frequencies; besides, both the rotary inertia and shear deformation should be negligible. Following the Bernoulli–Euler beam theory, the general equation for transverse vibrations of an elastic beam under axial compression and distributed transverse pressure is expressed by [1,3]

$$EIw^{\mathrm{IV}} + \rho A\ddot{w} + Fw'' = p(x) \tag{1}$$

where $p(x)$ is the distributed transverse pressure per unit axial length which is positive when it acts downward, F the compressive axial load which is positive in compression, w the transverse beam deflection which is positive if downward, I and A the moment of inertia of the beam cross-section and the cross-sectional area of the beam, and E and ρ, Young's modulus and the mass density. Thus, EI denotes the bending stiffness of the beam, and ρA represents the mass density per unit axial length. In addition, we define

$$w' = \frac{\partial w}{\partial x}, \quad \dot{w} = \frac{\partial w}{\partial t} \tag{2}$$

where x is the axial coordinate and t the time.

Eq. (1) can be applied to each of the beams of the elastically connected double-beam system shown in Fig. 1. Assuming that the two beams have the same effective material constants, it follows from Eq. (1) that

$$EI_1w_1^{\mathrm{IV}} + \rho A_1\ddot{w}_1 + F_1w_1'' + K(w_1 - w_2) = 0 \tag{3}$$

$$EI_2w_2^{\mathrm{IV}} + \rho A_2\ddot{w}_2 + F_2w_2'' + K(w_2 - w_1) = 0 \tag{4}$$

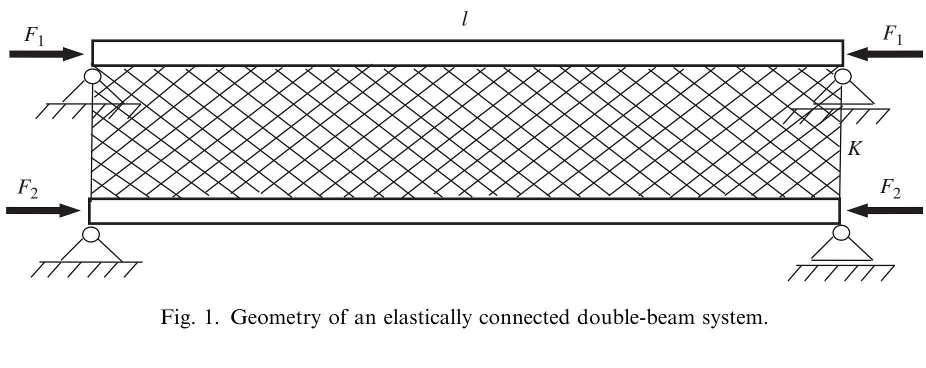

Fig. 1. Geometry of an elastically connected double-beam system.

where K is the stiffness modulus of a Winkler elastic layer. As can be seen, these two differential equations describe the free transverse vibrations of double-beam system under compressive axial load. When the effect of axial load is ignored, Eqs. (3) and (4) reduce to those vibrational equations given by Oniszczuk [17].

3. Solution of the problem

Suppose that both beams have the same length l of the elastically connected double-beam system and their ends are simply supported, the boundary conditions are given by

$$w_1(0,t) = w_1(l,t) = 0 \tag{5a}$$

$$w_1''(0,t) = w_1''(l,t) = 0 \tag{5b}$$

and

$$w_2(0,t) = w_2(l,t) = 0 \tag{6a}$$

$$w_2''(0,t) = w_2''(l,t) = 0 \tag{6b}$$

and the initial conditions are assumed as follows:

$$w_1(x,0) = w_{10}(x) \tag{7a}$$

$$w_2(x,0) = w_{20}(x) \tag{7b}$$

and

$$\dot{w}_1(x,0) = v_{10}(x) \tag{8a}$$

$$\dot{w}_2(x,0) = v_{20}(x) \tag{8b}$$

The homogeneous partial differential equations (3) and (4) with the governing boundary conditions (5) and (6) can be solved by the Bernoulli–Fourier method assuming the solutions in the form

$$w_1(x,t) = \sum_{n=1}^{\infty} X_n(x) T_{1n}(t) \tag{9}$$

$$w_2(x,t) = \sum_{n=1}^{\infty} X_n(x) T_{2n}(t) \tag{10}$$

where $T_{1n}(t)$ and $T_{2n}(t)$ denote the unknown time functions, and $X_n(x)$ is the known mode shape function for simply supported single beam, which is defined as

$$X_n(x) = \sin(k_n x) \tag{11}$$

with

$$k_n = \frac{n\pi}{l}, \quad n = 1, 2, 3, \ldots \tag{12}$$

Introduction of Eqs. (9) and (10) into Eqs. (3) and (4) yields

$$\sum_{n=1}^{\infty} (\rho A_1 \ddot{T}_{1n} + (EI_1 k_n^4 + K - F_1 k_n^2) T_{1n} - K T_{2n}) X_n = 0 \tag{13}$$

$$\sum_{n=1}^{\infty} (\rho A_2 \ddot{T}_{2n} + (EI_2 k_n^4 + K - F_2 k_n^2) T_{2n} - K T_{1n}) X_n = 0 \tag{14}$$

It follows from Eqs. (13) and (14) that a set of ordinary differential equations for the unknown time functions can be expressed as

$$\ddot{T}_{1n} + (N_1 - \eta_1 F_1) T_{1n} - H_1 T_{2n} = 0 \tag{15}$$

$$\ddot{T}_{2n} + (N_2 - \eta_2 F_2)T_{2n} - H_2 T_{1n} = 0 \tag{16}$$

with

$$N_1 = \frac{EI_1 k_n^4}{\rho A_1} + H_1, \quad N_2 = \frac{EI_2 k_n^4}{\rho A_2} + H_2 \tag{17}$$

$$\eta_1 = \frac{k_n^2}{\rho A_1}, \quad \eta_2 = \frac{k_n^2}{\rho A_2} \tag{18}$$

$$H_1 = \frac{K}{\rho A_1}, \quad H_2 = \frac{K}{\rho A_2} \tag{19}$$

The solutions of Eqs. (15) and (16) can be obtained by

$$T_{1n}(t) = C_n e^{j\omega_n t}, \quad T_{2n}(t) = D_n e^{j\omega_n t}, \quad j = \sqrt{-1} \tag{20}$$

where ω_n denotes the natural frequency of the double-beam system, and C_n and D_n represent the amplitude coefficients of the two beams, respectively. Substituting Eq. (20) into Eqs. (15) and (16), we obtain

$$(N_1 - \eta_1 F_1 - \omega_n^2)C_n - H_1 D_n = 0 \tag{21}$$

$$(N_2 - \eta_2 F_2 - \omega_n^2)D_n - H_2 C_n = 0 \tag{22}$$

When the determinant of the coefficients in Eqs. (21) and (22) vanishes, non-trivial solutions for the constants C_n and D_n can be obtained, which yields the following frequency (characteristic) equation:

$$\omega_n^4 - (N_1 + N_2 - \eta_1 F_1 - \eta_2 F_2)\omega_n^2 + (N_1 - \eta_1 F_1)(N_2 - \eta_2 F_2) - H_1 H_2 = 0 \tag{23}$$

It can be observed that the discriminant of this biquadratic algebraic equation is positive

$$\Delta = (N_1 - N_2 - \eta_1 F_1 + \eta_2 F_2)^2 + 4H_1 H_2 > 0 \tag{24}$$

Then from the characteristic Eq. (23), we obtain

$$\omega_{nI}^2 = \tfrac{1}{2}\left(N_1 + N_2 - \eta_1 F_1 - \eta_2 F_2 - \sqrt{(N_1 - N_2 - \eta_1 F_1 + \eta_2 F_2)^2 + 4H_1 H_2}\right) \tag{25}$$

$$\omega_{nII}^2 = \tfrac{1}{2}\left(N_1 + N_2 - \eta_1 F_1 - \eta_2 F_2 + \sqrt{(N_1 - N_2 - \eta_1 F_1 + \eta_2 F_2)^2 + 4H_1 H_2}\right) \tag{26}$$

where ω_{nI} is the lower natural frequency of the system, and ω_{nII} is the higher natural frequency. For each of the natural frequencies, the associated amplitude ratio of vibration modes of the two beams is given by

$$\alpha_n = \frac{C_n}{D_n} = \frac{H_1}{N_1 - \eta_1 F_1 - \omega_n^2} = \frac{N_2 - \eta_2 F_2 - \omega_n^2}{H_2} \tag{27}$$

It is seen that without the axial load the results for ω_{nI}, ω_{nII}, and α_{nI} obtained by Oniszczuk [17] are recovered. Introducing Eqs. (25) and (26) into Eq. (27), respectively, we have

$$\alpha_{nI} = \frac{1}{2H_2}\left(N_2 - N_1 + \eta_1 F_1 - \eta_2 F_2 + \sqrt{(N_1 - N_2 - \eta_1 F_1 + \eta_2 F_2)^2 + 4H_1 H_2}\right) \tag{28}$$

$$\alpha_{nII} = \frac{1}{2H_2}\left(N_2 - N_1 + \eta_1 F_1 - \eta_2 F_2 - \sqrt{(N_1 - N_2 - \eta_1 F_1 + \eta_2 F_2)^2 + 4H_1 H_2}\right) \tag{29}$$

As can be seen, the amplitude ratio α_{nI} corresponding to the lower natural frequency ω_{nI} is always positive, which indicates that the two beams execute synchronous vibrations, while α_{nII} corresponding to the higher frequency ω_{nI} is always negative, which indicates that the two beams execute asynchronous vibrations.

From the above analysis we know that solutions (20) can be rewritten as

$$T_{1n}(t) = C_{1n} e^{j\omega_{nI} t} + C_{2n} e^{-j\omega_{nI} t} + C_{3n} e^{j\omega_{nII} t} + C_{4n} e^{-j\omega_{nII} t} \tag{30}$$

$$T_{2n}(t) = D_{1n}e^{j\omega_{nI}t} + D_{2n}e^{-j\omega_{nI}t} + D_{3n}e^{j\omega_{nII}t} + D_{4n}e^{-j\omega_{nII}t} \tag{31}$$

Substitution of the trigonometric functions into the above two equations gives

$$T_{1n}(t) = \sum_{i=\mathrm{I}}^{\mathrm{II}}(A_{ni}\sin(\omega_{ni}t) + B_{ni}\cos(\omega_{ni}t)) \tag{32}$$

$$T_{2n}(t) = \sum_{i=\mathrm{I}}^{\mathrm{II}}\alpha_{ni}(A_{ni}\sin(\omega_{ni}t) + B_{ni}\cos(\omega_{ni}t)) \tag{33}$$

where A_{ni} and B_{ni} ($i = \mathrm{I}$, II) are unknown constants which will be determined in the following. Then the transverse vibrations of the double-beam system under axial compressions can be described by

$$w_1(x,t) = \sum_{n=1}^{\infty}\sin(k_nx)\sum_{i=I}^{\mathrm{II}}(A_{ni}\sin(\omega_{ni}t) + B_{ni}\cos(\omega_{ni}t)) \tag{34}$$

$$w_2(x,t) = \sum_{n=1}^{\infty}\sin(k_nx)\sum_{i=\mathrm{I}}^{\mathrm{II}}\alpha_{ni}(A_{ni}\sin(\omega_{ni}t) + B_{ni}\cos(\omega_{ni}t)) \tag{35}$$

On the basis of the orthogonality property of mode shape functions, the unknown constants A_{ni} and B_{ni} can be determined from the assumed initial conditions (7) and (8). To find the final form of the transverse vibrations, the initial-value problem is solved. In this case, the classical orthogonality condition is applied:

$$\int_0^l X_mX_n\,dx = \int_0^l \sin(k_mx)\sin(k_nx)\,dx = \beta\delta_{mn} \tag{36}$$

with

$$\beta = \int_0^l X_n^2\,dx = 0.5l \tag{37}$$

where δ_{mn} is the Kronecker delta. Introduction of Eqs. (34) and (35) into the initial conditions (7) and (8) yields

$$w_{10} = \sum_{n=1}^{\infty}\sin(k_nx)\sum_{i=\mathrm{I}}^{\mathrm{II}}B_{ni} \tag{38}$$

$$v_{10} = \sum_{n=1}^{\infty}\sin(k_nx)\sum_{i=\mathrm{I}}^{\mathrm{II}}\omega_{ni}A_{ni} \tag{39}$$

and

$$w_{20} = \sum_{n=1}^{\infty}\sin(k_nx)\sum_{i=\mathrm{I}}^{\mathrm{II}}\alpha_{ni}B_{ni} \tag{40}$$

$$v_{20} = \sum_{n=1}^{\infty}\sin(k_nx)\sum_{i=\mathrm{I}}^{\mathrm{II}}\alpha_{ni}\omega_{ni}A_{ni} \tag{41}$$

Multiplying the above equations by the eigenfunction X_m, integrating them with respect to x from 0 to l, and using the orthogonality condition (36), we have

$$\beta^{-1}\int_0^l w_{10}\sin(k_nx)\,dx = \sum_{i=\mathrm{I}}^{\mathrm{II}}B_{ni} \tag{42}$$

$$\beta^{-1}\int_0^l v_{10}\,\sin(k_n x)\,\mathrm{d}x = \sum_{i=\mathrm{I}}^{\mathrm{II}} \omega_{ni}A_{ni} \tag{43}$$

$$\beta^{-1}\int_0^l w_{20}\,\sin(k_n x)\,\mathrm{d}x = \sum_{i=\mathrm{I}}^{\mathrm{II}} \alpha_{ni}B_{ni} \tag{44}$$

$$\beta^{-1}\int_0^l v_{20}\,\sin(k_n x)\,\mathrm{d}x = \sum_{i=\mathrm{I}}^{\mathrm{II}} \alpha_{ni}\omega_{ni}A_{ni} \tag{45}$$

It follows from the above equations that

$$A_{n\mathrm{I}} = \frac{1}{\beta(\alpha_{n\mathrm{II}}-\alpha_{n\mathrm{I}})\omega_{n\mathrm{I}}}\int_0^l (\alpha_{n\mathrm{II}}v_{10}-v_{20})\,sin(k_n x)\,\mathrm{d}x \tag{46}$$

$$A_{n\mathrm{II}} = \frac{1}{\beta(\alpha_{n\mathrm{I}}-\alpha_{n\mathrm{II}})\omega_{n\mathrm{II}}}\int_0^l (\alpha_{n\mathrm{I}}v_{10}-v_{20})\,\sin(k_n x)\,\mathrm{d}x \tag{47}$$

and

$$B_{n\mathrm{I}} = \frac{1}{\beta(\alpha_{n\mathrm{II}}-\alpha_{n\mathrm{I}})}\int_0^l (\alpha_{n\mathrm{II}}w_{10}-w_{20})\sin(k_n x)\,\mathrm{d}x \tag{48}$$

$$B_{n\mathrm{II}} = \frac{1}{\beta(\alpha_{n\mathrm{I}}-\alpha_{n\mathrm{II}})}\int_0^l (\alpha_{n\mathrm{I}}w_{10}-w_{20})\sin(k_n x)\,\mathrm{d}x \tag{49}$$

4. Application

For simplicity, in what follows we assume that the two beams of the elastically connected double-beam system have the same bending stiffness and cross-sectional area. It follows from the assumption that

$$N_1 = N_2 = N = \frac{EIk_n^4}{\rho A} + H \tag{50}$$

$$\eta_1 = \eta_2 = \eta = \frac{k_n^2}{\rho A} \tag{51}$$

$$H_1 = H_2 = H = \frac{K}{\rho A} \tag{52}$$

The values for the parameters of the system which are used in the numerical calculations are given as follows:

$$E = 1\times 10^{10}\,\mathrm{N\,m^{-2}},\quad A = 5\times 10^{-2}\,\mathrm{m}^2,\quad I = 4\times 10^{-4}\,\mathrm{m}^4$$

$$K_0 = 2\times 10^{5}\,\mathrm{N\,m^{-2}},\quad l = 10\,\mathrm{m},\quad \rho = 2\times 10^{3}\,\mathrm{kg^{-3}}$$

4.1. The axial buckling load

When the natural frequency of the system vanishes under the axial loading, the system begins to buckle. Introduction of $\omega_n = 0$ into Eq. (23) and combination of Eqs. (50), (51), and (52) give

$$(N-\eta F_1)(N-\eta F_2) - H^2 = 0 \tag{53}$$

Without loss of generality, we assume

$$F_2 = \chi F_1 \tag{54}$$

with

$$0 \leqslant \chi \leqslant 1 \tag{55}$$

Thus, Eq. (53) can be rewritten as

$$\chi \eta^2 F_1^2 - (\chi + 1)\eta N F_1 + N^2 - H^2 = 0 \tag{56}$$

It follows from Eq. (56) that the value of the buckling stress corresponding to vibration mode n can be obtained by

$$(F_1)_b^I = \frac{(\chi + 1)\eta N + \sqrt{(\chi + 1)^2 \eta^2 N^2 - 4\chi \eta^2 (N^2 - H^2)}}{2\chi \eta^2} \tag{57}$$

and

$$(F_1)_b^{\mathrm{II}} = \frac{(\chi + 1)\eta N - \sqrt{(\chi + 1)^2 \eta^2 N^2 - 4\chi \eta^2 (N^2 - H^2)}}{2\chi \eta^2} \tag{58}$$

As can be seen, the values of the buckling loads $(F_1)_b^{\mathrm{I}}$ and $(F_1)_b^{\mathrm{II}}$ are both positive and $(F_1)_b^{\mathrm{I}} > (F_1)_b^{\mathrm{II}}$. Consequently, $(F_1)_b^{\mathrm{II}}$ is the critical buckling load corresponding to vibration mode n, namely

$$F_b^{\mathrm{cr}} = \frac{(\chi + 1)\eta N - \sqrt{(\chi + 1)^2 \eta^2 N^2 - 4\chi \eta^2 (N^2 - H^2)}}{2\chi \eta^2} \tag{59}$$

Assuming $\chi = 0$ and $K = 0$, from Eq. (56) we obtain

$$P_n = EI \frac{n^2 \pi^2}{l^2} \tag{60}$$

which is the critical buckling load corresponding the number n of the Euler beam. By setting n equal to 1, Eq. (60) reduces to

$$P = EI \frac{\pi^2}{l^2} \tag{61}$$

This load is known as the Euler load, which is the smallest load at which the beam ceases to be in stable equilibrium.

It can be observed from Eq. (59) that the critical buckling load F_b^{cr} corresponding to vibration mode n is dependent on the ratio χ of the axial load F_2 to F_1. With the stiffness modulus $K = K_0$, the influence of χ on the critical buckling load F_b^{cr} is shown in Fig. 2. It is seen that the critical buckling load F_b^{cr} decreases with increasing the axial compression ratio χ, and the effect of the axial load ratio χ is related to the vibration mode n. The larger the vibration mode n, the more significant the effect of the axial compression ratio on the critical buckling load. In addition, the critical buckling load F_b^{cr} is also dependent on the stiffness modulus K of the Winkler elastic layer. With $n = 3$, the effect of the Winkler elastic layer on the critical buckling load F_b^{cr} is shown in Fig. 3. It is indicated that the critical buckling load F_b^{cr} increases with the increase of the stiffness modulus K.

4.2. The effect of compressive axial loading

In the following, let us first illustrate the effect of compressive axial loading on the natural frequencies of transverse vibration of the double-beam system. When the axial compressions are absent, it follows from Eqs. (25), (26), (50)–(52) that

$$(\omega_{n\mathrm{I}}^0)^2 = N - H \tag{62}$$

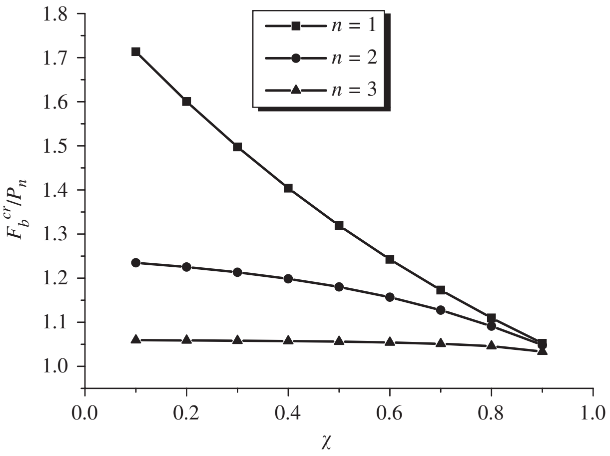

Fig. 2. Effect of the axial load ratio λ on the critical buckling load F_b^{cr}

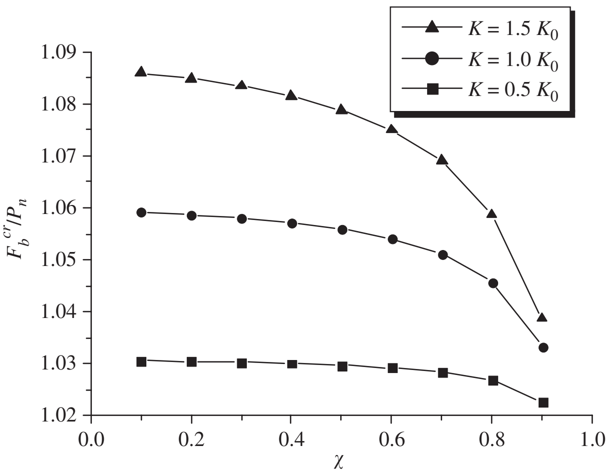

Fig. 3. Effect of the stiffness modulus K of the Winkler elastic layer on the critical buckling load F_b^{cr}.

$$(\omega_{n\text{II}}^0)^2 = N + H \tag{63}$$

where $\omega_{n\text{I}}^0$ and $\omega_{n\text{II}}^0$, respectively, denote the lower and higher natural frequencies of the system without axial loading. To examine the influence of compressive axial loading on the natural frequencies of transverse vibration of the system, the results of natural frequencies under compressive axial loading and those without axial loading are compared. It follows that

$$\omega_{n\text{I}}^2 = \tfrac{1}{2}(2N - \eta(1+\chi)\lambda F_b^{\text{cr}} - \sqrt{\eta^2\lambda^2(F_b^{\text{cr}})^2(1-\chi)^2 + 4H^2}) \tag{64}$$

$$\omega_{n\text{II}}^2 = \tfrac{1}{2}\left(2N - \eta(1+\chi)\lambda F_b^{\text{cr}} + \sqrt{\eta^2\lambda^2(F_b^{\text{cr}})^2(1-\chi)^2 + 4H^2}\right) \tag{65}$$

with

$$\lambda = \frac{F_1}{F_b^{\text{cr}}} \tag{66}$$

If we define

$$\psi_{\mathrm{I}} = \frac{\omega_{n\mathrm{I}}}{\omega_{n\mathrm{I}}^0}, \quad \psi_{\mathrm{II}} = \frac{\omega_{n\mathrm{II}}}{\omega_{n\mathrm{II}}^0} \tag{67}$$

with the vibration mode number $n = 3$ the effects of compressive axial loading on the natural frequencies of transverse vibration of the system represented by the ratios of ψ_{I} and ψ_{II} are shown in Figs. 4 and 5, respectively. As can be seen, the ratios ψ_{I} and ψ_{II} diminish with increasing the axial compression, which implies that the natural frequencies $\omega_{n\mathrm{I}}$ and $\omega_{n\mathrm{II}}$ become smaller when the axial loads get larger. Moreover, the natural frequencies of the system become more sensitive to the compressive axial loading as the critical buckling loads are approached. It can also be found from these two figures that the effect of compressive axial loading on the lower natural frequency $\omega_{n\mathrm{I}}$ is almost independent of the axial compression ratio χ whereas that on the higher natural frequency $\omega_{n\mathrm{II}}$ is significantly dependent on it. The increase of the axial compression ratio χ brings about an evident reduction of the higher natural frequency $\omega_{n\mathrm{II}}$.

In order to investigate the effect of compressive axial loading on the amplitude ratios of the two beams of the system, the results for the amplitude ratios under compressive axial loading and those without axial loading are compared. Without axial loading, it follows from Eqs. (28), (29), (50)–(52) that

$$\alpha_{n\mathrm{I}}^0 = 1, \quad \alpha_{n\mathrm{II}}^0 = -1 \tag{68}$$

where $\alpha_{n\mathrm{I}}^0$ and $\alpha_{n\mathrm{II}}^0$ denote the amplitude ratios of the two beams dependent on the lower and higher natural frequencies of the system without axial loading, respectively. With compressive axial loading, we have

$$\alpha_{n\mathrm{I}} = \frac{1}{2H}\left(\lambda\eta(1-\chi)F_b^{\mathrm{cr}} + \sqrt{\left(\lambda\eta(1-\chi)F_b^{\mathrm{cr}}\right)^2 + 4H^2}\right) \tag{69}$$

$$\alpha_{n\mathrm{II}} = \frac{1}{2H}\left(\lambda\eta(1-\chi)F_b^{\mathrm{cr}} - \sqrt{\left(\lambda\eta(1-\chi)F_b^{\mathrm{cr}}\right)^2 + 4H^2}\right) \tag{70}$$

If we define

$$\varphi_{\mathrm{I}} = \frac{\alpha_{n\mathrm{I}}}{\alpha_{n\mathrm{I}}^0}, \quad \varphi_{\mathrm{II}} = \frac{\alpha_{n\mathrm{II}}}{\alpha_{n\mathrm{II}}^0} \tag{71}$$

with the vibration mode number $n = 3$ the impacts of compressive axial loading on the amplitude ratios of the two beams of the system represented by the parameters φ_{I} and φ_{II} are shown in Figs. 6 and 7, respectively. From Fig. 6, it is seen that the ratio φ_{I} increases with increasing the axial compression and decreasing the axial compression ratio χ, which implies that the amplitude ratio $\alpha_{n\mathrm{I}}$ dependent on the lower natural frequency $\omega_{n\mathrm{I}}$

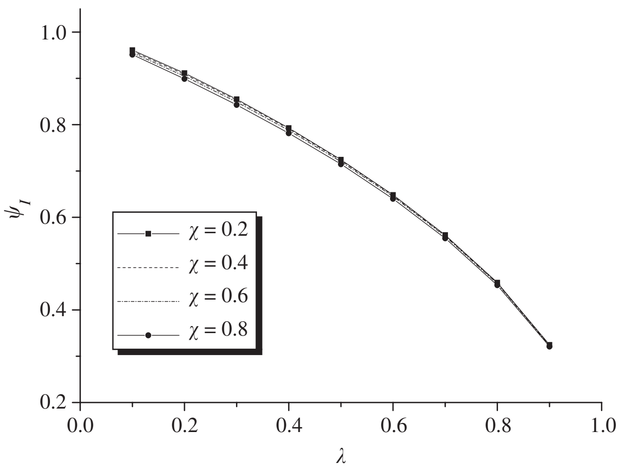

Fig. 4. Relationship between ratio $\psi_{\mathrm{I}} = \omega_{n\mathrm{I}}/\omega_{n\mathrm{I}}^0$ and dimensionless parameter $\lambda = F_1/F_b^{\mathrm{cr}}$.

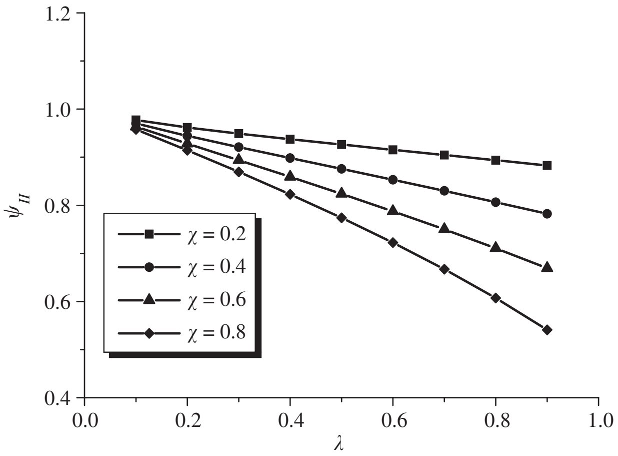

Fig. 5. Relationship between ratio $\psi_{\mathrm{II}} = \omega_{n\mathrm{II}}/\omega^0_{n\mathrm{II}}$ and dimensionless parameter $\lambda = F_1/F_b^{\mathrm{cr}}$.

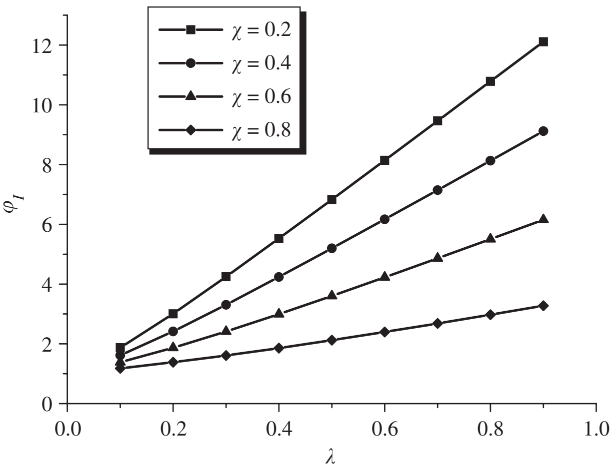

Fig. 6. Relationship between ratio $\varphi_{\mathrm{I}} = \alpha_{n\mathrm{I}}/\alpha^0_{n\mathrm{I}}$ and dimensionless parameter $\lambda = F_1/F_b^{\mathrm{cr}}$.

become larger when the axial compression gets larger and the axial compression ratio χ becomes smaller. From Fig. 7, it is found that the ratio φ_{II} diminishes with the increase of the axial compression and the decrease of the axial compression ratio χ. In other words, the increase of the axial compression ratio χ causes the reduction of the absolute value of the amplitude ratio $\alpha_{n\mathrm{II}}$ dependent on the higher natural frequency $\omega_{n\mathrm{II}}$ as it is negative. Consequently, it is concluded that the influences of compressive axial loading on the amplitude ratios $\alpha_{n\mathrm{I}}$ and $\alpha_{n\mathrm{II}}$ of the system become more significant with the increase of axial compression.

5. Conclusions

Based on the Bernoulli–Euler beam theory, the properties of free transverse vibration and buckling of an elastically connected simply supported double-beam system under compressive axial loading are studied. Using the classical Bernoulli–Fourier method, the solutions of the differential equations of motion for the system are formulated. The analytical solution for the critical buckling load of the system is derived. The explicit expressions are presented for natural frequencies and the associated amplitude ratios of the two beams. The effects of compressive axial loading on the responses of the double-beam system are investigated.

It is concluded that the critical buckling load is influenced by the ratio χ of the axial load F_2 to F_1 and the Winkler elastic layer. It gets smaller with the increase of the ratio χ and the diminishment of the stiffness

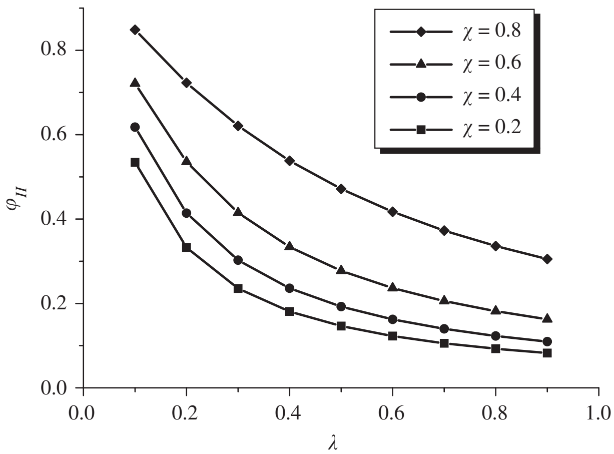

Fig. 7. Relationship between ratio $\varphi_{\mathrm{II}} = \alpha_{n\mathrm{II}}/\alpha^0_{n\mathrm{II}}$ and dimensionless parameter $\lambda = F_1/F_b^{\mathrm{cr}}$.

modulus K of the Winkler elastic layer. In addition, it is found that the effects of compressive axial loading on the natural frequencies of the system and associated amplitude ratios are more significant with the increase of axial compression. Moreover, the effects of compressive axial loading on the higher natural frequency and the amplitude ratios are significantly dependent on the axial compression ratio whereas that on the lower natural frequency is almost independent of it.

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